JOURNAL OF APPROXIMATION THEORY **39**, 72–76 (1983)

Proximity Maps and Fixed Points*

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Communicated by E. W. Cheney

Received February 26, 1982

In [4], Ky Fan proved the following result.

THEOREM 1. *Let C be a nonempty, compact convex subset of a normed linear space X. For any continuous function $f: C \to X$, there exists a y in C such that*

$$\|y - f(y)\| = d(f(y), C).$$

There have appeared several extensions and applications of this theorem. For example, it has been applied to the proof of Schauder's fixed point theorem. The aim of this note is to give a result related to the above with applications to fixed point theory in Hilbert space.

Let X be a normed linear space and C a nonempty subset of X. For $x \in X$, define

$$d(x, C) = \inf_{y \in C} \|x - y\|$$

and

$$P_C(x) = \{y \in C \mid \|x - y\| = d(x, C)\}.$$

The set-valued map, $P_C(x)$, is called the metric projection on C. In the case when C is a Chebyshev set, $P_C(x)$ is a point-map $X \to C$ called a proximity map.

If the compact set is replaced by a closed unit ball then Theorem 1 is no longer valid even in a Hilbert space. To see this consider the Hilbert space l^2 with unit ball B centred at the origin with $f: B \to B$ defined by

$$f(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, ..., x_n, ...).$$

* Supported in part by a NATO grant and NSERC Grant A-5154 of the first author, and NSERC Grant A-4646 of the second.

0021-9045/83 $3.00

Then f is continuous with $\|f\| = 1$. If there exists $y \in B$ with $\|y - f(y)\| = d(f(y), B)$ then f has a fixed point. But it is easy to see that f is fixed point free.

S. Reich (in [7]) extended the result in the following form.

THEOREM 2. *Let C be a closed, convex subset of a Banach space X having the Oshman property. For any continuous function $f: C \to X$ with $f(C)$ relatively compact, there exists a y in C such that*

$$\|y - f(y)\| = d(f(y), C).$$

E. V. Oshman (in [6]) has characterized those reflexive Banach spaces in which the metric projection on every closed, convex subset is upper semicontinuous. Theorem 2 above was used to get Schauder's fixed point theorem of the second form as an easy corollary.

Recently in [5] Lin gave the following version of the result for densifying maps.

THEOREM 3. *Let B_r be the closed ball (about the origin) with radius r in a Banach space X. If $f: B_r \to X$ is a continuous, densifying map, then there exists a y in B_r such that*

$$\|y - f(y)\| = d(f(y), B_r).$$

We need the following preliminary definitions and results. Let C be a convex subset of a Hilbert space H. The point $b \in C$ is the nearest point to $a \notin C$ if and only if $(x - b, b - a) \geqslant 0$ for all $x \in C$. If, additionally, C is closed, then $P_C(x)$, the proximity map, is nonexpansive. That is,

$$\|P_C(x) - P_C(y)\| \leqslant \|x - y\| \qquad \text{for all} \quad x, y \in H.$$

For details of the above see [3].

THEOREM 4. *Let B be a closed, bounded convex subset of a Hilbert space H with $f: B \to B$ nonexpansive. Then f has a fixed point.*

For the above result see [1].

We prove the following result.

THEOREM 5. *Let C be a closed, convex subset of a Hilbert space H and $f: C \to H$ nonexpansive with $f(C)$ bounded. Then there exists a y in C such that*

$$\|y - f(y)\| = d(f(y), C).$$

Proof. Let $P: H \to C$ be the proximity map. Then since both P and f are nonexpansive the map $Pf: C \to C$ is nonexpansive. Let B represent the convex closure of $Pf(C)$. Now, $Pf(C)$ is bounded since $f(C)$ is bounded. Hence, by Theorem 4, the map $Pf: B \to B$ has a fixed point $y \in B$. Therefore,

$$\|y - f(y)\| = \|Pf(y) - f(y)\| = d(f(y), C).$$

This completes the proof.

COROLLARY OF THEOREM 5. *If C is a closed, bounded, convex subset of a Hilbert space H and $f: C \to H$ is nonexpansive, then there exists a y in C such that*

$$\|y - f(y)\| = d(f(y), C).$$

As an application of Theorem 5, we give a short proof of the following result of Schöneberg (see [8]).

THEOREM 6. *Let H be a Hilbert space and C a closed, bounded, convex subset of H. Let $f: C \to H$ be a nonexpansive map such that for each x on the boundary of C,*

$$\|f(x) - y\| \leqslant \|x - y\| \qquad \text{for some } y \text{ in } C.$$

Then f has a fixed point.

Proof. By the Corollary to Theorem 5, there is a y_0 in C such that

$$\|y_0 - f(y_0)\| = d(f(y_0), C).$$

If $f(y_0) \in C$ then, clearly, $f(y_0) = y_0$ and we are finished. So suppose $f(y_0) \notin C$. In this case, $Pf(y_0) \in C$ and, by definition, $\|f(y_0) - Pf(y_0)\| = d(f(y_0), C)$. But this implies that $Pf(y_0) = y_0$ since the nearest point is unique. Furthermore, by hypothesis, there is a y in C such that $\|f(y_0) - y\| \leqslant \|y_0 - y\|$. It suffices to show $y = y_0$. But if this is not the case, then we have $\|y_0 - f(y_0)\| < \|f(y_0) - y\| \leqslant \|y_0 - y\|$. This implies that there is a point $y_1 = \alpha y + (1 - \alpha) y_0$ with $0 < |\alpha| < 1$ for which $\|y_1 - f(y_0)\| < \|y_0 - f(y_0)\|$. This is a contradiction. Hence $y_0 = y$. This completes the proof.

The following result of Browder and Petryshyn (see [2]) also follows easily from Theorem 5.

THEOREM 7. *Let H be a Hilbert space and C a closed, bounded, convex subset of H. Let $f: C \to H$ be nonexpansive and assume for any u on the* *boundary of C with $u = Pf(u)$ that is a fixed point of f. Then f has a fixed point.*

Proof. By the Corollary to Theorem 5, there is a y_0 in C such that $\|y_0 - f(y_0)\| = d(f(y_0), C)$. If $f(y_0) \in C$ then, clearly, $f(y_0) = y_0$. Otherwise, $Pf(y_0) \in C$ and $\|f(y_0) - Pf(y_0)\| = d(f(y_0), C)$. Hence $Pf(y_0) = y_0$ since the nearest point is unique. If y_0 is on the boundary, then $f(y_0) = y_0$ by hypothesis. Otherwise, $f(y_0)$ is in C in which case $y_0 = Pf(y_0) = f(y_0)$.

That Theorem 5 may yield a result when Browder's Theorem (Theorem 4) does not apply is seen by the following example.

Let $C = [0, \infty)$ and define $f: C \to R$ by $f(x) = 1/(1 + x)$. Then f is nonexpansive with $f(C)$ bounded. Using Theorem 5, it is easily seen that f has a fixed point which is $(\sqrt{5} - 1)/2$.

The following result follows easily from the Corollary to Theorem 5.

THEOREM 8. *Let B_r be the closed ball of radius r and centre 0 in the Hilbert space H. Let $f: B_r \to H$ be a nonexpansive map with the property that*

if $f(x) = \alpha x$ for some x on the boundary of B_r then $\alpha \leqslant 1$.

Then f has a fixed point.

Proof. By the Corollary of Theorem 5, there exists a $y_0 \in B_r$ such that $\|y_0 - f(y_0)\| = d(f(y_0), B_r)$. If $f(y_0) \in B_r$ we are done, so suppose that $\|f(y_0)\| > r$. By convexity, y_0 is on the boundary of B_r and $f(y_0) = \alpha y_0$ for some $\alpha > 0$. This gives

$$\alpha = \frac{\|\alpha y_0\|}{\|y_0\|} = \frac{\|f(y_0)\|}{\|y_0\|} > \frac{r}{r} = 1.$$

But this is a contradiction since, by hypothesis, $\alpha \leqslant 1$.

This completes the proof.

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